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EQUATIONS OF MOTION

OF A

VISCOUS LIQUID

BY

PEARL EUGENE DOUDNA, B. A. M.

A thesis presented for the degree of

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Historical Introduction

Two and a half centuries B. C. Archimedes (287?-212) wrote a work entitled DE IIS QUAE VEHUNTUR IN HUMIDO. He maintained that every particle of a fluid mass, when in equilibrium, is equally pressed in every direction. The laws and properties of liquids were investigated by this ancient mathematician sufficiently to enable him to devise a hydrostatic method for determining the purity of the precious metals. He made a further practical application of the results of his studies in this direction by the invention of the screw engine, or the Archimedean screw.

The Alexandrian School is accredited with the construction of a few hydraulic machines such as the siphon and the force pump. However fluid motion was probably first studied by a Roman, Sextius Julius Frontinus, inspector of the public fountains at Rome in the reigns of Nerva and Trajan about the close of the first century A. D. In his work entitled DE AQUAEDUCTIBUS URBIS ROMAE COMMENTARIUS, is found a description

of the great aqueducts of Rome. Frontinus constructed five new aqueducts, making in all fourteen. He also describes the methods used in determining the amount of water discharged from ajutages and the methods of distributing the water of the aqueducts and fountains. He observed that the amount of water discharged depends upon the height of the water in the reservoir above the orifice as well as upon the area of the orifice.

Fifteen centuries later Castelli (1628) advanced the theory that the velovity of discharge is proportional to the height of water in the reservoir above the orifice. Torricelli, a contemporary of Castelli, observing that a small jet of water rushing from an ajutage rises to a height almost equal to that of the water in the reservoir, concluded that the velocity of discharge is equal to the velocity which the water would acquire by falling freely through a distance the same as the height of water above the orifice. As a result

of this observation he obtained the correct relation between the velocity of discharge and the head of water (1643).

Mariotte (1620?-1684), the author of a posthumous work, entitled TRAITE DU MOUVEMENT DES EAUX, made extensive use of the theorem of Torricelli. He seems to have been the first to attempt to reconcile theory and experiment by attributing the retardation to friction. The filiments sliding along the surface of the pipe were supposed to be retarded and other filiments having a greater velocity than those naer the surface were retarded by rubbing against the slower ones. The retardation was supposed to be proportional to the distance from the axis of the pipe.

Guglielmini, a contemporary of Mariotte, devoted himself to the study of the motiom of the water in rivers and canals. He assumed that every particle in a verticle (sic) section moves with a velocity equal to the velocity of discharge from an orifice at an equal depth below the surface of the water, and

the column of the fluid having the area of the orifice as a base. Therefore this weight which is proportional to the area of the orifice multiplied by the height of fluid in the vessel must be proportional to the quantity of motion engendered in the particles which escape through the orifice each instant. Continuing this analysis he finally arrives at the theorem already established by Torricelli[1] by an entirely different method.

The investigation of the flow of water in rivers attracted considerable attention in Italy, probably on account of the extensive landscape gardening. Besides Guglielmini, who was inspector of rivers in the Milanese, Marquis Poleni deserves mention in this connection. In 1695 he wrote DE MOTU AQUAE MIXTO; and in 1718 another work concerning the flow of water through orifices and short tubes.

Newton (1642-1727) investigated the effects of friction and viscosity in diminishing the velocity of running water.

In book 2,§9,of the PRINCIPIA he offers the following hypothesis: The resistance which arises from the viscosity of a fluid,other things being equal,is proportional to the velocity with which the particles separate from each other. This may be said to be the fundamental princip~~al~~le underlying all that part of hydrodynamics which deals with viscous fluids. The VENA CONTRACTA as well as the oscillation of waves seems to have been considered first by Newton.

Daniel Bernoulli published his HYDRODYNAMICA,SIVI DE.VIRIBUS ET MOTIBUS FLUIDORUM COMMENTARII,in 1738,in which he bases his theory upon the suppositions that the surface of a fluid contained in a vessel which is being emptied by an orifice remains always horizontal and that horizontal strata always remain contiguous to each other,and that the particles descend vertically with avelocity inversly proportional to the horizontal section of the reservoir. His principle was not acceptable to his contemporaries,consequently John Ber-

noulli and Maclaurin each attempted to solve the problem by independent methods but without marked success.

Jean-le-Rond D'Alembert (1717-1783) aided by the discoveries of Euler (1707-1783) took the first great step in determining the general equations of motion of a perfect fluid. "When generalizing James Bernoulli's theory of pendulums he discovered a principle of dynamics so simple and general that it reduced the laws of the motions of bodies to that of their equilibrium. He applied this principle to the motion of fluids,and gave a specimen of its application at the end of his Dynamics in 1743. It was more fully developed in his TRAITE DES FLUIDES,which was published in 1744,where he resolves,in the most simple and ellegant manner,all the problems which relate to the equilibrium and motion of fluids. He makes use of the very same suppositions as Daniel Bernoulli,though his calculus is established in a very different manner. He considers,at every instant,the actual motion of a stratum as com-

posed of a motion which it had in the preceeding instant and of a motion which it has lost. The laws of equilibrium between the motions lost furnish him with the equations which represent the motion of the fluid. Although the science of hydrodynamics had then made considerable progress, yet it was chiefly founded on hypothesis. It remained a desideratum to express by equations the motion of a particle of a fluid in any assigned direction. These equations were found by D'Almbert from two principles: 1st that a rectangular canal, taken in a mass of fluid in equilibrium, is itself in equilibrium; 2d and that a portion of fluid, in passing from one place to another, preserves the same volume when the fluid is incompressible, or dilates itself according to a given law when the fluid is elastic. His very ingenious method was published in 1752, in his ESSAI SUR LA RESISTANCE DES FLUIDES. It was brought to perfection in his OPUSCULAR MATHEMATIQUES, and was adopted by Euler." Philosophers had attempted in vain to de-

termine the laws of fluid motion independent of all hypotheses. However the method of fluxions proved inadequate to the purpose and it was only after Euler had contributed to science his calculus of partial differences that the object was reached. D'Alembert first applied the new calculus to the motion of water, and he and Euler both succeeded in obtaining equations of motion for a perfect fluid restricted by no particular hypothesis.

Chevalier Dubuat, a French engineer, published in 1786 a very satisfactory theory of the motion of a fluid, based upon the
experiments
experience of himself and others. "He considered that if water werea perfect fluid and the channels in which it flows infinitely smooth, its motion would be continnually accelerated, like that of bodies descending on inclined planes. But since the motion of rivers is not accelerated, and soon arrives at a state of uniformity, it is evident that the viscosity of the water and the friction of the channel in which it

descends must equal the accelerating force. Dubuat therefore assumes it as a proposition of fundamental importance, when water flows in any channel or bed, that the accelerating force which obliges it to move is equal to all the resistances which it meets with, whether they arise from the viscosity or from the friction of the bed."

In 1779 Abbé Bossut conducted very extensive investigations to determine the amount of retardation in pipes due to adhesion and friction. He reached the conclusion that with pipes of the same length the discharge is proportional to the diameters, and when the diameters are equal the discharge is inversly proportional to the square root of the lengths.

The general equations of motion are so complex that little could be done with them. This resulted in the numerous efforts which were made by one class of investigators such as Dubuat, Possut, and Prony to develop by experiment simpler formulae for the velocity of running water. However the more

theoretical mathematicians, among whom were Lagrange, Laplace, and Poisson, endeavored to obtain solutions of the equations of D'Alembert and thus solve the various problems of hydrodynamics. These men together with other eminent mathematicians both of France and Great Britain devoted much time to the investigation of waves and tides.

Navier (1785-1836) was probably the first to offer equations of motion for a viscous fluid (1822). He bases his investigation upon the following suppositions:-the fluid is composed of ultimate molecules, acting upon each other by forces which, when the fluid is at rest, are functions simply of the distance, but when the molecules recede from or approach each other, are modified by this circumstance, so that two molecules repel each other less strongly when they are receding and more strongly when they are approaching each other than when at rest; and the alteration in the attraction or repulsion is supposed to be proportional to the velocity with which the

molecules, separated by a given distance, are approaching or receding from each other: a second supposition involves the symetrical arrangement of the particles. Navier dealt only with an incompressible fluid and arrived at the following equations: $X-1/\rho(dp/dx-A(d^2u/dx^2+d^2u/dy^2+d^2u/dz^2))=\partial u/\partial t$ and the other two can be written by making similar changes in the corresponding equations for a perfect fluid. A is a constant depending on the nature of the fluid. The other terms are the same as those used in the following pages.

Poisson (1781-1840) derived equations not only for an incompressible fluid but also for an elastic fluid in which the change in density is small. He treated the subject from the standpoint of an elastic solid, supposing the fluid to be continually beginning to be displaced like an elastic solid, and continually re-arranging itself so as to make the presure equal in all directions as is the case with a fluid at rest. His equations are writtenthus

$X - 1/\rho(dp/dx - A(d^2u/dx^2 + d^2u/dy^2 + d^2u/dz^2) - B\ d/dx(du/dx + dv/dy + dw/dz))$ $= \partial u/\partial t$ with the other two to correspond.

For an incompressible fluid these agree with those obtained by Navier since in that case the expression for cubical expansion disappears. It is to be noticed that both of the foregoing methods involve a consideration of the ultimate molecules of the fluid.

Barre de Saint-Venant (17~~79~~97-1886) first succeeded in obtaining the equations independent of any consideration of the ultimate molecules. He attempts to connect the oblique pressures in different directions about a point with the differential coefficients du/dx, du/dy &c, which express the relative motion of the fluid particles in the immediate neighborhood of the point, by assuming the tangential force on a plane passing through the point to be in the direction of the principal sliding along that plane. He then employes theorems by Cauchy, connecting the pressures in ~~any~~ different directions in any

fluid,to show that the tangential force in any direction along a plane is proportional to the sliding along the same plane resolved in the same direction. He published this theory in 1843.

In 1845 G. G. Stokes derived the same equations as Poisson but found that A=3P. He based his investigation upon th three principal hypotheses:1stThat the difference between the pressures on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P;that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned".(Stokes) He then arrives at the conclusion that the stresses due to viscosity are functions of the rates of strain. After speculating as to the molecular constitution of the flu

id he arrives at the hypothesis that these are linear functions. This is his second hypothesis. For gases he introduces a third:"When a gas is expanding equally in all directions the stresses P,Q,and R are the same as if the fluid were frictionless....." In his report to the British Association in 1846 ON RECENT RESEARCHES IN HYDRODYNAMICS,Stokes claims that the principal feature of his investigation consists in eliminating from the relative motion of a fluid about,any particular point the relative motion which corresponds to a certain motion of rotation,and examining the nature of the relative motion which remains;and that the method employed does not necessarily require the consideration of the ultimate molecules.

The second assumption of Stokes is not altogether satisfactory since it rests upon the supposition that the velocity is small.

In 1861 C. E. Meyer derived the ordinary equations if F=0 which,according to other investigators prior to this time,

are true only for an incompressible fluid. He begins his investigation after some preliminary considerations with the following statement:"The internal friction of a fluid takes place between the different strata of the same fluid,and is proportional to the differential-coefficient of the velocity along the normal to the plane separating the strata." But this hypothesis explains only six of his nine initial expressions. For instance if we take the three initial expressions for the retarding forces parallel to the x-axis

$$-n(du/dx)dy\ dz,$$

$$-n(du/dy)dz\ dx,$$

$$-n(du/dz)dx\ dy,$$

The second and third readily follow from the hypothesis but the first has no definite meaning,unless we extend his hypothesis as W. M. Hicks has interpreted it in the British Association report for 1881 so that it reads-"The friction on a small plane in a given direction in the plane is proportional

to the rate of variation perpendicular to the plane of the component of the velocityin the given direction, whilst there is a normal part, proportional to the rate of variation perpendicular to the plane, of the component perpendicular to this plane."

The subject has been investigated by Stefan (1862) and later by Maxwell, M. Levy, M. Kleitz, and Butcher without adding materially to what had been done already.

However the work of Maxwell resulted in the determinationof the constant called the coefficient of viscosity. Following Maxwell numerous experiments have been made for the purpose of determining the value of the coefficient of viscosity. Helmholtz, Piotrowski, Maxwell, Meyer, and Poisseuille may be named as having made the most elaborate series of experiments.

Lamb, in his TREATISE ON FLUIDS (1879), derives the equations of motion for a viscous fluid by a method based on those of St.

Venant and Stokes. Basset, in 1888, published A TREATISE ON HYDRODYNAMICS in which he uses Stokes' method. Both authors accept the second hypothesis of Stokes so that we have no assurance that their equations hold except in the case of slow motion. Both fully relized this and restricted the application of their results accordingly. Of all methods so far that of Stokes had received the most general approval.

In the following paper we have derived the equations for the motion of a perfect fluid in order that we might extend the same methods as far as possible in obtaining the equations for a viscous liquid (the discussion is limited to an incompressible fluid, or liquid). The second part is devoted to the consideration of a viscous liquid. The discussion is based upon the definition of the coefficient of viscosity as agreed upon by experimental physicists. Then by comparing the motion of a viscous with that of a perfect liquid we are enabled to give a definite meaning to all nine of

the initial expressions obtained by Meyer. To obtain the equations referred to cylindrical and spherical coordinates we have not used the method of transformation but have derived them by analysis similar to that used in determining the equations when referred to rectangular coordinates. By this method a definite meaning attaches to each term in the resulting equations.

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Colorado Springs, May 11, 1897.

PART I

EQUATIONS OF MOTION OF A PERFECT FLUID

D'Alembert's Principle: Since D'Alembert's principle is used so frequently in what follows, it may be well to say by way of explanation of this principle that its use means the equating of

Impressed Force - Pressure - Expressed Force to the forces lost through friction.

An Ideal Fluid: With this as with other problems it is easier to consider first the ideal condition. An ideal fluid is one in which there is no friction; the particles move by one another without causing any mutual retardation; hence without any loss of energy.

The Derivation of the Equations of Motion of a Perfect Fluid: To derive the equations of motion of a perfect fluid is a comparatively easy matter. Let A D (Fig.1) represent an element of such a fluid; x, y, and z be the coordinates of

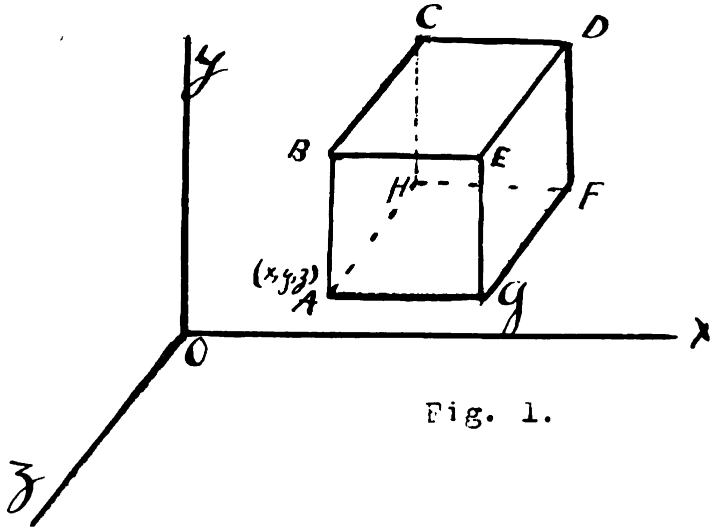

Fig. 1.

the point A;A G,A F,and A H be represented by dx,dy,and dz respectively;X?Y,andZ the impressed force resolved parallel to the x,y,and z axes;u,v,and w the components of the velocity in the same directions;and finally let p represent the pressure at the point (x,y,z) or A and m the mass of the element.

From the conditions of the problem

$$u = f(x,y,z,t)$$

and it follows immediately that

$$\partial u/\partial t = du/dt + u(du/dx)+ v(du/dy)+ w(du/dz)= f_x$$

This last is an expression for the effective force or expressed force in the direction of the x-axis. In a similar no-

tation $\partial v/\partial t$, or f_y and $\partial w/\partial t$, or f_z represent the effective forces in the direction of the y and z-axes respectively. The total impressed forces are mX, mY, and mZ; the total effective forces mf_x, mf_y, and mf_z; the pressures on the faces A C, A E, and A F are p dy dz, p dy dx, and p dx dz respectively; on the faces opposite the pressures are p dy dz + (dp/dx)dx dy dz, p dy dx + (dp/dz)dx dy dz, and p dx dz + (dp/dy)dx dy dz. Therefore the excess of pressures on the opposite faces are

$$(dp/dx)dx\ dy\ dz,\ (dp/dz)dx\ dy\ dz,\ \text{and}\ (dp/dy)dx\ dy\ dz.$$

Since the hypothesis excludes the possibility of any loss through friction, we have now considered all the forces involved. Equating separately the forces acting in the direction of each axis by D'Alembert's principle, we obtain the following equations:

$$mX - mf_x - (dp/dx)dx\ dy\ dz = 0$$

$$mY - mf_y - (dp/dy)dx\ dy\ dz = 0$$

$$mZ - mf_z - (dp/dz)dx\ dy\ dz = 0$$

Substituting for m its value $\rho\, dx\, dy\, dz$ in which ρ represents the density of the fluid, and then dividing by $dx\, dy\, dz$, the resulting equations are:

$$X\rho - f_x\rho - dp/dx = 0$$

$$Y\rho - f_y\rho - dp/dy = 0$$

$$Z\rho - f_z\rho - dp/dz = 0$$

or as usually written:

$$X - 1/\rho \; dp/dx = \partial u/\partial t$$

$$Y - 1/\rho \; dp/dy = \partial v/\partial t$$

$$Z - 1/\rho \; dp/dz = \partial w/\partial t$$

So far we have obtained three relations between the five quantities u, v, w, p, and ρ. When the equations are used in application to an incompressible fluid ρ is constant; but when other fluids are under consideration, a fourth equation is supplied by Boyle's Law,

$$p = k\rho$$

Continuity: The condition of continuity furnishes a

fifth equation. If the fluid be continuous, then the increase within a given space during any given time must equal the excess of inflow over the outflow across the boundaries in the given time.

The mass of fluid contained by the element at any moment is

$$\rho\, dx\ dy\ dz$$

and its increase in the time dt is

$$(d\rho/dt)dt\ dy\ dx\ dz$$

The flow over the surface dy dz in time dt is

$$u\rho\ dt\ dy\ dz$$

and the flow over the opposite face is

$$u\rho\ dt\ dy\ dz + d(u\rho)/dx\ \ dt\ dx\ dy\ dz,$$

giving an increase in the element of

$$(d(u\rho)/dx)dt\ dx\ dy\ dz.$$

Similarily the increments in the directions of the y and z-axes are found to be

$$(d(v\rho)/dy)dt\ dx\ dy\ dz$$

and

$$(d(w\rho)/dz)dt\ dx\ dy\ dz.$$

Equating the whole increment to the partial increments and dividing by $dt\ dx\ dy\ dz$ we obtain

$$d\rho/dt = d(u\rho)/dx + d(v\rho)/dy + d(w\rho)/dz.$$

But for an incompressible fluid, ρ being a constant, this becomes

$$du/dx + dv/dy + dw/dz = 0.$$

Cylindrical Coordinates: The equations of motion of a perfect fluid when referred to cylindrical coordinates may be derived from those already obtained for rectangular coordinates by transformation. The following however is a more satisfactory method in so much as it gives a definite interpretation to each term appearing in the resulting equations. Moreover this method to be applied in the derivation of the equations when referred to cylindrical and polar coordinates

forms the basis of several important principles involved in the derivation of the equations of motion of a viscous liquid referred to the same systems of coordinates.

The velocities u,v,and w,and the impressed forces X,Y, and Z have the directions of r, θ ,and z respectively. v is angular velocity.

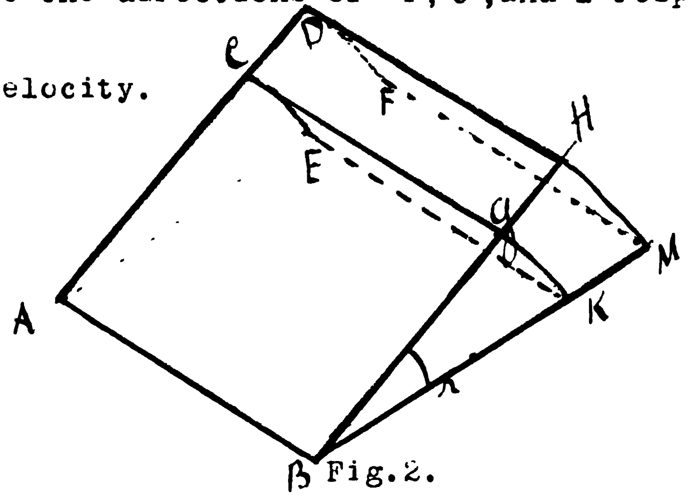

Fig.2.

$X\rho\ r\ d\theta\ dz\ dr$ = the impressed force in the direction of r.

$(\partial u/\partial t)\rho r\ d\theta\ dz\ dr$ = efective force in the same direction due to an acceleration in the velocity u. The velocity v through centrifugal force contributes to the effective force in this direction

$$-v^2\rho r^2\ d\theta\ dz\ dr.$$

The velocity w can have no influence upon the forces in the

direction of r.

$$-(dp/dr)r\,d\theta\,dz\,dr$$

is the difference in pressure on the two faces C K and D M.

Equating by D'Alembert's principle we have

$$X\rho\,r\,d\theta\,dz\,dr - (dp/dr)r\,d\theta\,dz\,dr - (\partial u/\partial t)\rho\,r\,d\theta\,dz\,dr + v^2\rho r^2 d\theta\,dz\,dr = 0$$

Dividing by $\rho\,r\,d\theta\,dz\,dr$ and transposing, this equation becomes

$$X - \frac{1}{\rho}dp/dr = \partial u/\partial t - v^2 r.$$

Similarily for the forces acting in the direction of θ

$Y\rho\,r\,d\theta\,dz\,dr$ + impressed force,

$r(\partial v/\partial t)\rho\,r\,d\theta\,dz\,dr$ = effective force contributed by acceleration in v.

$(1/r)(dp/d\theta)r\,d\theta\,dz\,dr$ = difference in the pressures on the opposite faces. But there is still another resulting force acting in this direction due to the fact that while a particle is being carried in the direction θ it is

~~it is~~ also being carried in the direction of r by a velocity u; consequently at the end of the time dt it will have fallen behind by an amount equal to

$$(r + dr)d\theta \ - r\ d\theta \ \text{ or } dr\ d\theta,$$

which may be written

$$\frac{r\ d\theta \ \ dr}{r.}$$

From physics

$$s = 1/2 \ \ ft^2$$

$$f = (2s)/t^2$$

where f = acceleration, and s the space described and t the time. Substituting

$$f = (2\ dr\ d\theta\ )/dt^2 = 2uv$$

which is the force due to the fall.

Multiplying by $\rho\, r\ d\theta\ \ dr\ dz$ we get for the total of this force

$$2\ u\ v\rho\, r\ d\theta\ \ dr\ dz.$$

The velocity along z as before contributes no component to

the forces in this direction.

Equating the forces found and simplifying we get

$$Y - (1/\rho)(1/r)(\partial p/\partial\theta) = r(\partial v/\partial t) + 2uv.$$

Neither u nor v can contribute a component to the forces acting in the direction of the z-axis, therefore the equation of these forces will be identical with the corresponding equation for rectangular coordinates.

Collecting the results just obtained we have

$$X - (1/\rho)(\partial p/\partial r) = \partial u/\partial t - v^2 r,$$

$$Y - (1/r)(1/\rho)(\partial p/\partial\theta) = r(\partial v/\partial t) + 2uv,$$

$$Z - (1/\rho)(\partial p/\partial z) = \partial w/\partial t.$$

The equation of continuity is

$$\partial(ur)/\partial r + r(\partial v/\partial\theta) + r(\partial w/\partial z) = 0.$$

It is unnecessary to derive this last equation since the method by which it is obtained is identical with that used to obtain the similar equation referred to rectangular coordinates.

Spherical Coordinates: Angle WCF (Fig.3.) = θ and

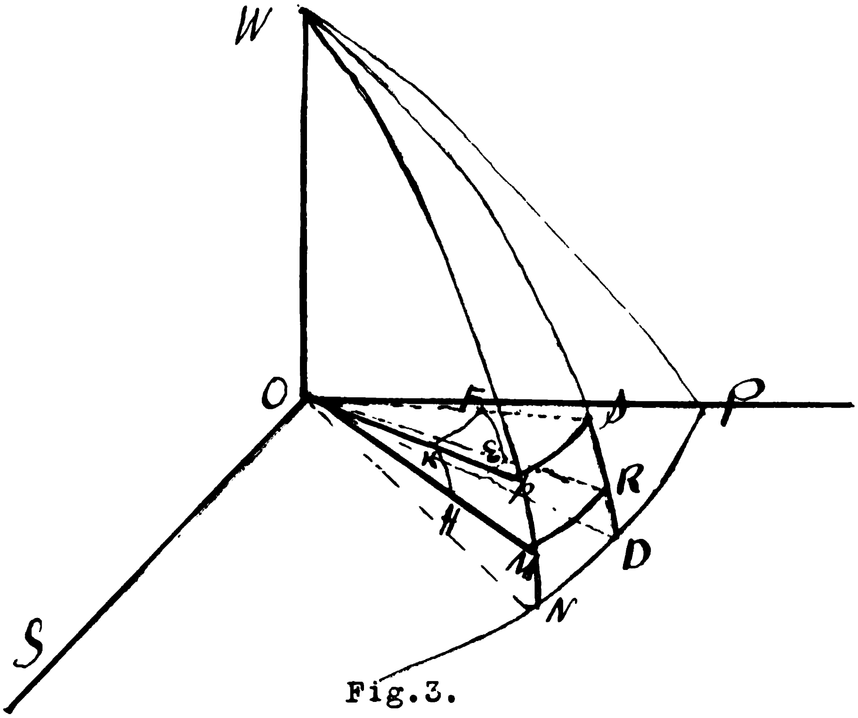

Fig.3.

PCD = Φ. Let u,v,and w represent the velocities in the directions r, θ, and Φ respectively; X,Y, and Z are symbols for the corresponding impressed forces. v and w are angular velocities. The mass of the element F M is

$$\rho \, dr \; r d\theta \; r \sin e \theta \; d\Phi = m.$$

$$Xm = \text{impressed force}$$

in the direction of r;

$$(\partial u/\partial t) \; m = \text{effective force}$$

due to the acceleration in u; the centrifugal force

$$-r \; v^2 m$$

results from the velocity v; the centrifugal force contributed by w is

$$-r \text{ sine } \theta \ w^2 m$$

in the direction of O D and this gives the component

$$-r \text{ sine}^2 \theta \ w^2 m$$

along the radius vector r;

$$(dp/dr)dr \ rd\theta \ \ r \text{ sine } \theta \ d\phi =$$

the difference in pressure on the two opposite faces.

Equating and transforming as before

$$X - (1/\rho)(dp/dr) = \partial u/\partial t - r \ v^2 - r \text{ sine}^2 \theta \ w^2.$$

With reference to the direction of   we have

$$Y \ m = \text{impressed force},$$

$$r(\partial v/\partial t) \ m = \text{difference in pressure}$$

on the opposite faces.

Since as a particle moves from e to F it is carried by u the distance e R it falls behind by an amount equal to $dr \ d\theta$ which when expressed in terms of force is equivalent to

$$2uv$$

or multiplying by mass

$$2uvm$$

represents the component of effective force contributed by the combined action of u and v. w cannot exert an effective force such as this since the particle always moves in the same parallel, but it does exert a centrifugal force in the direction of θ, or rather there is an effective component of the centrifugal force acting in the direction of θ due to w. We already know (see page 31) that the centrifugal force due to w along O D is

$$-r \text{ sine } \theta \ w^2.$$

Since the tangent to e F at e makes an angle θ with C D, the component of

$$- r \text{ sine } \theta \ w^2$$

along the tangent is

$$-r \text{ sine } \theta \cos \theta \ w^2.$$

Multiplying by mass we have

$$-r \text{ sine } \theta \cos \theta \; w^2 m.$$

Equating and simplifying, the resulting equation is

$$Y - (1/\rho)(1/r)(dp/d\theta) = r(\partial v/\partial t) + 2uv - r \text{ sine } \theta \cos \theta \; w^2$$

The forces along the Φ coordinate are

$$Z\,m = \text{impressed force}$$

$$\text{sine } \theta \, (\partial w/\partial t) r\, m = \text{effective force}$$

due to the acceleration of the angular velocity w;

$$dp/(\rho r \text{ sine } \theta \, d\Phi)\, m = \text{difference in the}$$

pressure on the opposite faces.

Since by the combined action of u and w a particle falls behind

$$dr \text{ sine } \theta \, d\Phi$$

which, converted into the terms of force, gives

$$2 \text{ sine } \theta \, w \, u.$$

Therefore we obtain a total force contributed by the combined action of these two velocities equal to

$$2 \text{ sine}\,\theta\; w\; u\; m;$$

v also carries the particle through the distance e F and since the line F K is shorter than e H there will be a resulting effective force due to the combined action of v and w.

$$e\,H = r \text{ sine}\,\theta\, d\phi$$

$$F\,K = r \text{ sine}(\theta + d\theta)\, d\phi \quad \text{(see note p 36)}$$

$$= r\, d\phi\, (\text{sine}\,\theta \cos d\theta + \cos\theta \sin d\theta).$$

Substituting 1 for $\cos d\theta$ and $d\theta$ for sine $d\theta$

$$F\,K = r\, d\phi \text{ sine}\,\theta + r\, d\phi \cos\theta\, d\theta$$

$$F\,K - e\,H = r\, d\phi \cos\theta\, d\theta$$

or the distance through which the particle may be said to fall under the combined action of w and v. This is equivalent to a force

$$2\, r \cos\theta\; w\; v\;,$$

or for the mass of the element

$$2\, r \cos\theta\; w\; v\; m.$$

By equating and simplifying we obtain the third equation

$Z - dp/(\rho r \ \text{sine}\,\theta \ d\phi) = r \ \text{sine}\,\theta(\partial w/\partial t) \ 2 \ \text{sine}\,\theta \ u \ w + 2rvw \cos\theta$

Equation of continuity: The increase in the element for the time dt is

$$(d\rho/dt)r^2 \ \text{sine}\,\theta \ d\phi \ dr \ d\theta;$$

the difference between the inflow and the outflow in the direction of r is

$$(d(ur^2\rho)/dr)\text{sine}\,\theta \ d\phi \ dr \ d\theta;$$

in the direction of θ

$$(d(rv\rho \ \text{sine}\,\theta)/(r \ d\theta))r^2 d\phi \ d\theta \ dr;$$

and in the direction of ϕ

$$(d(r \ \text{sine}\,\theta \ w\rho)/(r \ \text{sine}\,\theta \ d\phi))r^2 \ \text{sine}\,\theta \ d\phi \ d\theta \ dr.$$

Equating and dividing by $r^2 d\phi \ d\theta dr$

$$(d\rho/dt)\text{sine}\,\theta = (d(ur^2\rho)/(r \ dr))\text{sine}\,\theta + d(rv\rho \ \text{sine}\,\theta)/(r \ d\theta) +$$

$$(d(r \ \text{sine}\,\theta \ w \rho)/(r \ \text{sine}\,\theta \ d\phi))\text{sine}\,\theta$$

or

$$(d\rho/dt)\text{sine}\,\theta = (d(ur^2\rho)/(r^2dr))\text{sine}\,\theta + d(v\rho \ \text{sine}\,\theta)/d\theta +$$

$$(d(w\rho)/d\phi)\text{sine}\,\theta$$

or if ρ is a constant

$$(d(ur^2)/(r^2 dr))\text{sine}\,\theta + d(v\ \text{sine}\,\theta)/d\theta + (dw/d\phi)\,\text{sine}\,\theta = 0.$$

Collecting the equations obtained we have

$$X - dp/(\rho\, dr) = \partial u/\partial t - r\, v^2 - r\ \text{sine}^2\theta\, w^2,$$

$$Y - dp/(\rho\, r\, d\theta) = r(\partial v/\partial t) + 2uv - r\ \text{sine}\,\theta\ \cos\theta\ w^2,$$

$$Z - dp/(\rho\, r\ \text{sine}\,\theta\, d\phi) = r\ \text{sine}\,\theta\ (\partial w/\partial t) + 2\ \text{sine}\,\theta\, uw + 2r\ \cos\theta\ vw,$$

$$(d(ur^2)/(r^2 dr))\text{sine} + d(v\ \text{sine}\,\theta)/d\theta + (dw/d\phi)\text{sine}\,\theta = 0.$$

NOTE; On page 34 we say $F\,K = r\ \text{sine}(\theta + d\theta)\ d\phi$. It is to be noticed that $d\theta$ is taken in the negative sense.

PART II.

THE EQUATIONS OF MOTION OF A VISCOUS LIQUID

Having determined the equations of motion of a perfect fluid which is incompressible or compressible when referred to the principal systems of axes, we will now proceed to the consideration of an incompressible viscous fluid which we shall call a liquid. In part I we have said little about a compressible fluid in the derivation of the equations but a little consideration will show that the same form for the three principal equations holds true for both kinds of fluids. However in what follows we shall have occasion to refer to the foregoing equations as applying to a perfect liquid only. The problem we are about to consider involves the forces already dealt with in the case of a perfect liquid together with the forces lost through internal friction.

The problem is to determine the relations existing between the forces lost and the velocities.

Coefficient of Viscosity: It is necessary at this point to define a quantity called the coefficient of viscosity.

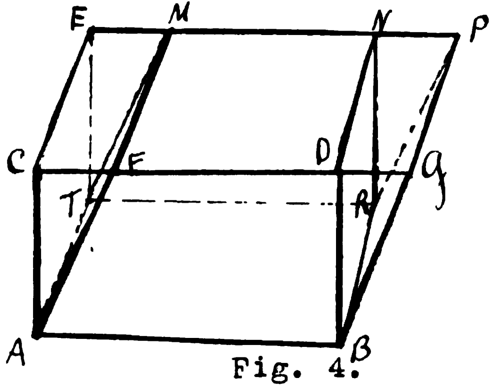

Fig. 4.

If A E D R represents the initial position of an element subjected to a shearing force exerted parallel to the planes of the strata e.g.plane A T R B such that the element takes the form A M G R, the coefficient of viscosity is defined as the ratio of the shearing force to the shear per unit of time. Putting this definition in the form of an equation we have

$$(\text{shearing force})/(\text{shear per unit of time}) = \mu$$

in which μ is the coefficient of viscosity:or shearing force $= \mu$ times shear per unit of time.

Let u be the velocity in the direction of A B; and the point A be located by x,y,z; and the sides of the element be dx dy dz. Then

$$u = f(x,y,z)$$

$$du/dy = f'(x,y,z) =$$

rate of normal displacement along y. This force acts upon the surface of the element dx dz. Hence the shearing force on this surface is

$$F = \mu (du/dy) dx\, dz.$$

The force on the opposite face is

$$F' = \mu (du/dy) dx\, dz + \mu (d^2u/dy^2) dx\, dz\, dy.$$

The difference of the forces of the opposite faces of the element considered is the retarding force due to the viscosity of the liquid.

Derivation of the Equations: Sufficient has now been said to proceed to the derivation of the equations. We shall take up this problem in much the same way that we did the

problem of a perfect liquid. But this new element of viscosity must be taken into account. It is evident that our equations, when derived, must be the same as those already obtained for a perfect liquid with terms attached to account for the retardation due to viscosity. For if in our new equations μ be equated to zero the equations for a perfect liquid must remain.

Forces to be Equated: The forces to be equated are

Impressed force,

Pressure,

Effective force, and

Retarding force, or force lost,

which give when equated

Impressed force-Pressure+Effective= -Retarding force

or in the more usual form

Impressed force-Pressure=Effective force-Retarding force.

It is to be noticed that the sense in which the retarding

force is taken is opposite that in which the impressed force is taken.

Rectangular Coordinates:

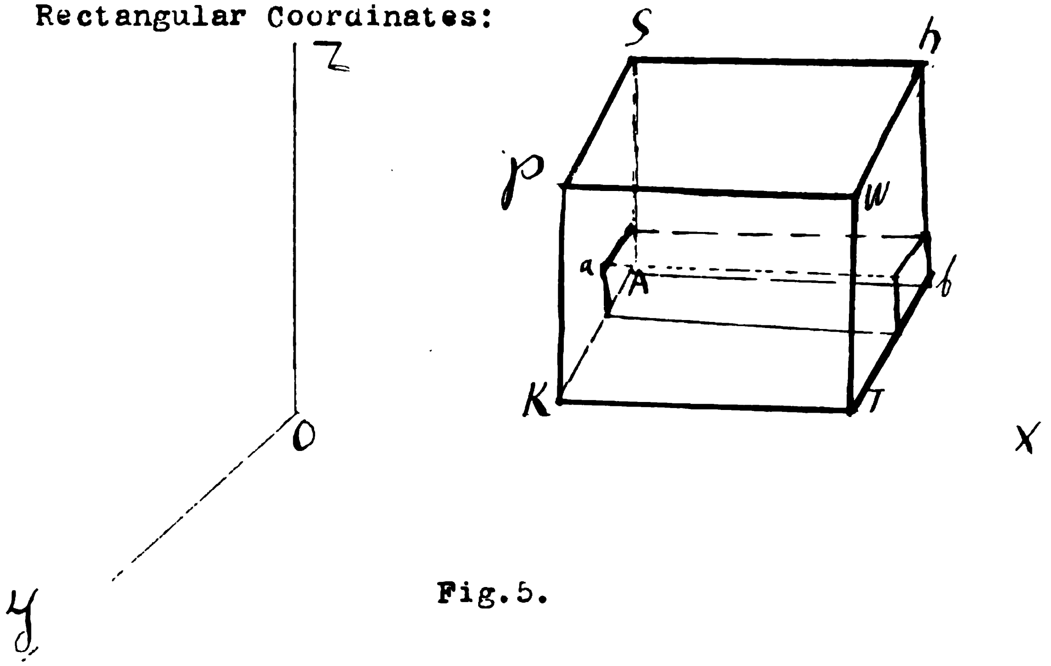

Fig.5.

Take an element p b,the coordinates of the point A being x,y, and z and the dimensions of the element being dx dy dz.

Suppose the velocity uniform in the direction of x.

Then any parallelopiped such as a b would move as a solid;but the element is composed of a great many such parallelopipeds. If the velocity in the direction of x varies as we pass from A to K,the layers in slipping by each other will produce - mutual retardation. The rate of change in u along the normal

to its direction is

$$du/dy.$$

From the definition of viscosity it follows that the shearing force upon the face s b is

$$\mu(du/dy)dx\ dz$$

and that upon the opposite face is

$$\mu(du/dy)dx\ dz + \mu(d^2u/dy^2)dx\ dy\ dz$$

indicating a loss of force within the element equal to

$$\mu(d^2u/dy^2)dx\ dz\ dy.$$

In a similar manner we obtain the retardation upon the face dx dy to be

$$\mu(du/dz)dx\ dy$$

and on the opposite face to be

$$\mu(du/dz)dx\ dy + \mu(d^2u/dz^2)dx\ dy\ dz$$

showing a loss within the element of

$$\mu(d^2u/dz^2)dx\ dy\ dz.$$

All of the loss within the element is now accounted for pro-

vided the velocity in the direction of x is uniform. It remains to examine the effect of an accelerating ~~force~~ velocity along x. To simplify the problem suppose the motion two dimen~~t~~sional. In this case it is only necessary to consider a section such as A C in Fig.6.

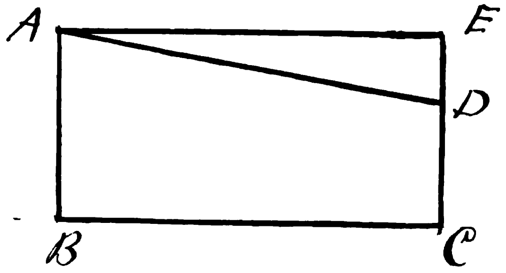

Fig.6.

Suppose un~~ear~~der certain conditions a certain motion exists at B If the liquid is perfect a certain amount will ~~of~~ flow into the parallelogram A C across the line A E due to the acceleration of the velocity u. E D represents the whole increase of inflow at E over that at A.

$$E D = A B - D C.$$

$$(A P)/ (D C) = u'/u$$

where u is the velocity across A B and u' that across D C.

By composition

$$(A\ B - D\ C)/A\ B = (u'-u)/u'$$

which represents the distance of inflow at A when a unit length of A B is considered. The time of inflow is dt hence

$$(A\ B - D\ C)/(A\ B\ dt) = (u' - u)/(u'dt)$$

gives the rate of displacement in the direction of x. Expanding

$$(1/dt)((u'-u)/u') = (1/dt)(1 - u/u') =$$

$$(1/dt)(1 - u/(u+du)) = (1/dt)(1-(1 - du/u + du^2/u^2 - du^3/u^3 + du^4/u^4 \text{ etc.}))$$

$$= (1/dt)(du/u - du^2/u^2 + du^3/u^3\ du^4/u^4 \text{ etc.})$$

Since this series converges very rapidly unless the increase in motion is very great all of the terms except the first may be dropped. This gives

$$du/(u\ dt) = du/dx$$

The retarding force on the face dy dz is

$$\mu (du/dx) dy\ dz$$

and on the opposite face

$$\mu\,(du/dx)\ dy\ dz + \mu\,(d^2u/dx^2)\,dy\ dz\ dx,$$

showing a loss for the element of

$$\mu\,(d^2u/dx^2)\ dy\ dz\ dx$$

when the liquid is viscous.

The retardation in the direction of the x-axis must equal the retardation of inflow, for there would be discontinuity if it were otherwise.

It is necessary to extend this principle to the case where the inflow is across two faces instead of one.

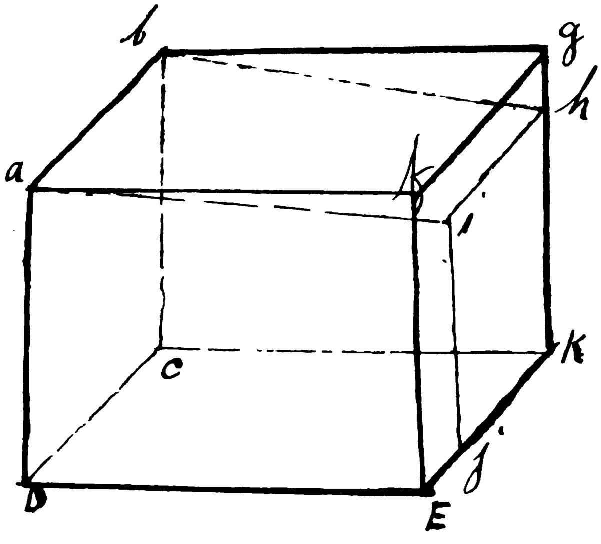

Fig. 7.

$$ab\ .\ ad - hi\ .\ ij =$$

total areal increase of inflow.

$$(ab\ ad)/(hi\ ij) = u'/u\ .$$

By composition

$$(ab\ ad - hi\ ij)/(ab\ ad\ dt) = (u'-u)/(u'dt)$$

equals rate of inflow. But

$$(u'-u)/(dt\ u') = (1/dt)(1 - u/(u + du)).$$

Expanding this and neglecting,as before all terms except the first we get

$$du/dx$$

for the rate of inflow. This shows that the lost forces obtained for two dimentional flow hold in the case when the flow is over two faces. This could have been concluded without the above proof for only so much can flow in and it is immaterial whether it all flows in the direction of z or partly in the direction of z ~~and partly in the direction of z~~ and partly in the direction of y.

The forces in the direction of the y and z axes can be obtained in the same way ~~that~~ those for the direction of x,

It is to be noticed that a force element of the second order is neglected if, for example, there is a normal increase in the velocity u, for in that case the inflow will be accelerated.

Equating the forces found and writing the two remaining symetrical equations we have

$$X - dp/(\rho\, dx) = \partial u/\partial t - \mu/\rho\,(d^2u/dx^2 + d^2u/dy^2 + d^2u/dz^2)$$

$$Y - dp/(\rho\, dy) = \partial v/\partial t - \mu/\rho\,(d^2v/dx^2 + d^2v/dy^2 + d^2v/dz^2)$$

$$Z - dp/(\rho\, dz) = \partial w/\partial t - \mu/\rho\,(d^2w/dx^2 + d^2w/dy^2 + d^2w/dz^2)$$

Cylindrical Coordinates:

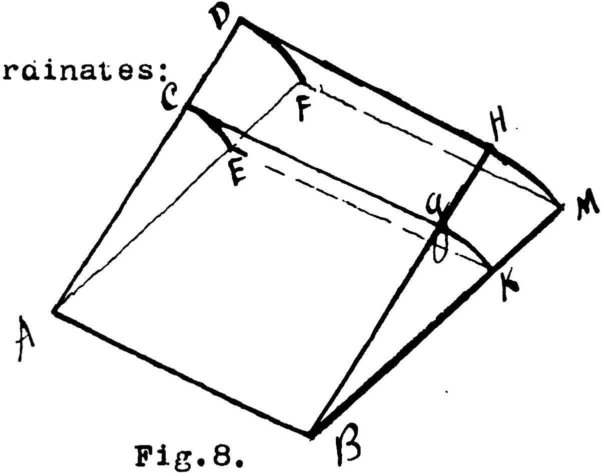

Fig.8.

In treating the equations for cylindrical and spherical coordinates it is found that the total rate of displacement as for example on the face C K along the normal is not ex-

pressed by du/dr for if the velocity u is constant there must be an inflow equivalent to u/r. Consequently in the following treatment account is taken of this and then the element considered as uniform in size i.e. $r\ d\theta\ dz$ is a constant so long as we are considering the acceleration in u along r.

With cylindrical coordinates for the two analogous cases, the acceleration of v along θ and w along z, the opposite faces are equal in size; but in spherical coordinates in two of the three cases, u along r and v along θ, the opposite faces are different. These two cases will be dealt with as the corresponding case in cylindrical coordinates.

One more consideration is necessary. It has been shown that v contributes an acceleration in the direction of r equal to

$$-r\ v^2.$$

If the change in v due to a change dt in the time is dv then the force

$$-r\,v^2$$

gives arate of displacement at the surface dr dz in the direction of θ equal to

$$-2\,v$$

i.e. angular displacement. In a similar manner we obtain two rates of displacement from

$$2\,uv$$

which contribute to the forces acting in the direction of θ viz.

$$2\,u/r$$

normal to the face dr dz and

$$2\,v/r$$

normal to the face $r\,d\theta\,dz$.

Now we may write the rates of displacement. The total rates normal to the three faces that contribute to the retarding forces in the direction r are

(1) $du/dr + u/r$ normal to $r\,d\theta\,dz$,

(2) $du/(r\,d\theta) - 2v$ normal to $dr\,dz$

(3) du/dz . normal to $r\,d\theta\,dr$.

The corresponding rates for the direction θ are

(4) $dv/dr + 2\,v/r$ normal to $r\,d\theta\,dz$,

(5) $dv/d\theta + 2\,u/r$ normal to $dr\,dz$,

(6) dv/dz normal to $r\,d\theta\,dr$.

And for the direction z we have

(7) dw/dr normal to $r\,d\theta\,dz$,

(8) $dw/(r\,d\theta)$ normal to $dr\,dz$,

(9) dw/dz normal to $r\,d\theta\,dr$.

Remembering that the surfaces over which rates (1),(5),and (9) act are to be considered constant--(two of them are actually constant)--and multiplying the rates by their respective surfaces and finding the difference in displacement for each opposite pair of faces we have the following nine differences corresponding to the nine initial expressions on pages 49 and 50

(1) $(d^2u/dr^2 + du/(r\ dr) - u/r^2)r\ d\ dz\ dr,$

(2) $(d^2u/(r^2\ d\theta^2) - (2dv)/(r\ d\theta))r\ d\theta\ dr\ dz,$

(3) $(d^2u/dz^2)r\ d\theta\ dr\ dz,$

(4) $(d^2v/dr^2 + (3\ dv)/(r\ dr))r\ d\theta\ dz\ dr,$

(5) $(d^2v/(r\ d\theta^2) + (2\ du)/(r^2 d\theta))r\ d\theta\ dz\ dr,$

(6) $(d^2v/dz^2)r\ d\theta\ dz\ dr,$

(7) $(d^2w/dr^2 + dw/(r\ dr))r\ d\theta\ dz\ dr,$

(8) $(d^2w/(r^2\ d\theta\ ))r\ d\theta\ dz\ dr,$

(9) $(d^2w/dz^2)r\ d\theta\ dz\ dr.$

The differences obtaihed from (4) and (6) are rates in change in angular displacement as compared with linear distances and consequently must be multiplied by the radius vector before being equated.

Multiplying these expressions by the coefficient of viscosity,and then adding them to the corresponding equations for a perfect liquid we have

$$X - dp/(\rho dr) = \partial u/\partial t - r\ v^2 - \nu\left(\frac{d^2u}{dr^2} + \frac{du}{r\,dr} - \frac{u}{r^2} + \frac{d^2u}{r^2 d\theta^2} - \frac{2dv}{r\,d\theta} + \frac{d^2u}{dz^2} \right)$$

$$Y - dp/(\rho r\, d\theta) = r\, \partial v/\partial t + 2uv - \nu\left(r \frac{d^2v}{dr^2} + 3\frac{dv}{dr} + \frac{d^2v}{r\,d\theta^2} + \frac{2\,du}{r^2 d\theta} + r\frac{d^2v}{dz^2}\right)$$

$$Z - dp/(\rho\, dz) = \partial w/\partial t - \nu\left(\frac{d^2w}{dr^2} + \frac{dw}{r\,dr} + \frac{d^2w}{r^2 d\theta^2} + \frac{d^2w}{dz^2}\right)$$

/The letter ν used in these equations equals μ/ρ the coefficient of viscosity divided by the density.

Spherical Coordinates:

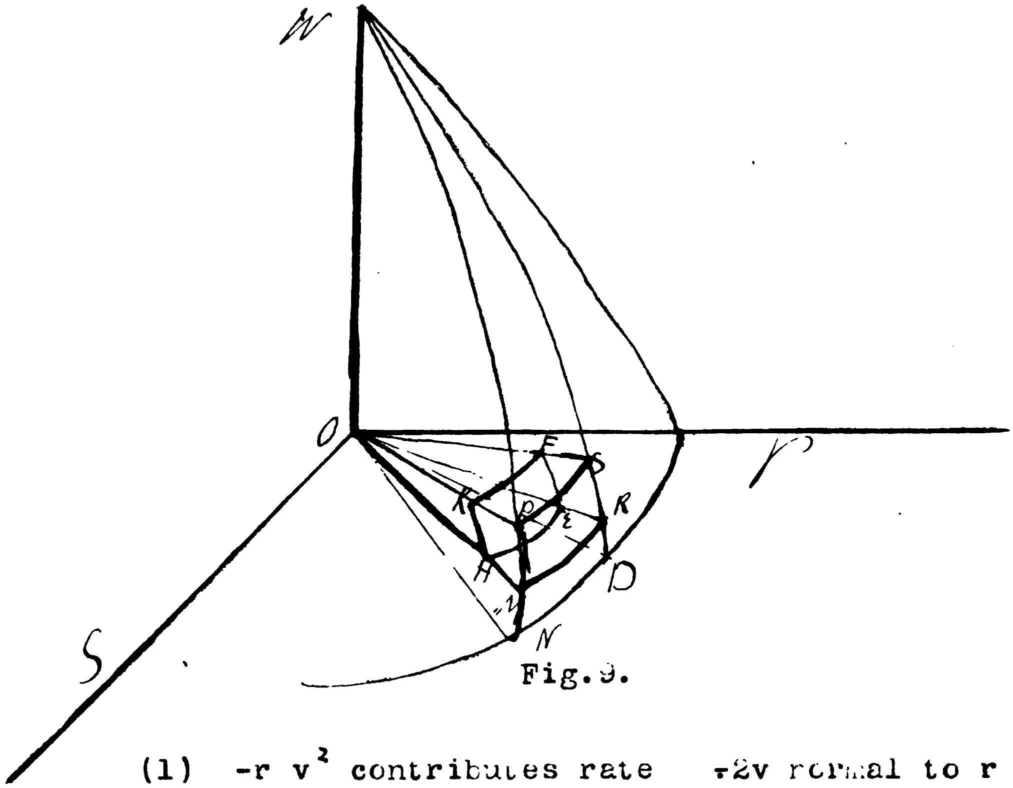

Fig. 9.

(1) $-r\, v^2$ contributes rate $+2v$ normal to r sine$\theta\, d\phi\, dr$

(2) $-r$ sine$^2\theta\, w^2$ " . . . " -2 sine $\theta\, w$ " $r\, d\theta\, dr$,

(3) $2uv$ ...."............$2u/r$./.......r sine θ $d\phi\, dr$,

(4) $2uv$..........................$2v/r$..........$r\, d\theta\, r\, d\phi$ sineθ,

(5) $-r$ sine θ cos $\theta\, w^2$..........2 cos$\theta\, w$ ....$r\, d\theta\, dr$,

(6) $2\ \text{sine}\,\theta\ w\ u$......$(2\text{sine}\,\theta\ w)/(r\ \text{sine}\,\theta)$... $r\ \text{sine}\,\theta\ d\phi\ r\ d\theta$,

(7) $2\ \text{sine}\,\theta\ w\ u$......$(2\text{sine}\,\theta\ u)/(r\ \text{sine}\,\theta)$... $r\ d\theta\ dr$,

(8) $2r\ \cos\theta\ v\ w$......$(2\ \cos\theta\ w)/\text{sine}\,\theta$ ......$r\ \text{sine}\,\theta\ d\phi\ dr$,

(9) $2r\ \cos\theta\ v\ w$......$(2\ \cos\theta\ v)/\text{sine}\,\theta$ ......$r\ d\theta\ dr$.

In taking account of the acceleration of u along r and of v along θ it must be remembered that if every thing is constant there is an inflow in the first case equivalent to

$$2u/r$$

and in the second case equivalent to

$$\cot\theta\ v.$$

In dealing with spherical coordinates we shall consider the element of uniform size in the same cases as in cylindrical coordinates. The rates of displacement are

(1) $du/dr + 2u/r$ on face $r^2\ \text{sine}\,\theta\ d\phi\ d\theta$,

(2) $du/(r\ d\theta) - 2v$.......$r\ \text{sine}\,\theta\ d\phi\ dr$,

(3) $du/(r\ \text{sine}\,\theta\ d\phi) - 2\text{sine}\,\theta\ w$......$r\ d\theta\ dr$,

(4) $dv/dr + 2v/r$........$r^2\ \text{sine}\,\theta\ d\phi\ d\theta$,

(

(5) $dv/d\theta + 2u/r + v\cot\theta$ on face $r\,\text{sine}\,\theta\,d\varphi\,dr$,

(6) $dv/(\text{sine}\,\theta\,d\varphi) - 2\cos\theta\,w$......$r\,d\theta\,dr$,

(7) $dw/dr + 2w/r$..................$r^2\,\text{sine}\,\theta\,d\varphi\,d\theta$,

(8) $dw/d\theta + 2\cot\theta\cdot w$...............$r\,\text{sine}\,\theta\,d\varphi\,dr$,

(9) $dw/d\varphi + 2u/r + 2\cot\theta\,v$........$r\,d\theta\,dr$.

Remembering that the surfaces upon which (1), (5), and (9) act are to be considered constant we obtain for the differences in displacement

(1) $(d^2u/dr^2 + 2du/(r\,dr) - 2u/r^2)r^2\,\text{sine}\,\theta\,d\varphi\,d\theta\,dr$,

(2) $(d^2u/(r^2 d\theta^2) + du/(r^2 d\theta)\cot\theta - 2dv/(r\,d\theta) - 2v/r\,\cot\theta)r^2\,\text{sine}\,\theta\,d\theta\,d\varphi\,dr$,

(3) $(d^2u/(r^2\,\text{sine}^2\theta\,d\varphi^2) - 2dw/(r\,d\varphi))r^2\,\text{sine}\,\theta\,d\varphi\,dr\,d\theta$,

(4) $(d^2v/dr^2 + 4dv/(r\,dr) + 2v/r^2)r^2\,d\theta\,d\varphi\,dr\,\text{sine}\,\theta$,

(5) $(d^2v/(r\,d\theta^2) + 2du/(r^2 d\theta) + dv/(r\,d\theta)\cot\theta - v/(r\,\text{sine}^2\theta))\,r^2\,\text{sine}\,\theta\,d\varphi\,dr\,d\theta$,

(6) $(d^2v/(r\,\text{sine}^2\theta\,d\varphi^2) - 2\cos\theta\,dw/(r\,\text{sine}\,\theta\,d\varphi))r^2\,\text{sine}\,\theta\,d\varphi\,dr\,d\theta$,

(7) $(d^2w/dr^2 + 4\,dw/(r\,dr) + 2w/r^2)r^2\,\text{sine}\,\theta\,d\varphi.dr\,d\theta$,

(8) $(d^2w/(r\,d\theta^2) + 3\cot\theta\,dw/(r\,d\theta) - 2w/r)r^2\,\text{sine}\,\theta\,d\varphi\,dr\,d\theta$,

(9) $(d^2w/(r\,\text{sine}\,\theta\,d\varphi^2) + 2\,du/(r^2\,\text{sine}\,\theta\,d\varphi) + 2\cot\theta\,dv/(r\,\text{sine}\,\theta\,d\varphi))\,r^2\,\text{sine}\,\theta\,d\varphi\,dr\,d\theta$,

The expressions derived from (4),(7), and (8) must be multiplied by the proper radii before equating.

Finally we have by annexing these quantities, multiplied by the coefficient of viscosity, to the corresponding equations for a perfect liquid

$X - dp/(\rho\,dr) = \partial u/\partial t - v^2 r - r\,\text{sine}^2\theta\, w^2 - \nu(d^2u/dr^2 + 2du/(r\,dr)$

$- 2u/r^2 + d^2u/(r^2 d\theta^2) + \cot\theta\, du/(r^2 d\theta) - 2dv/(r\,d\theta) - \cot\theta\, 2v/r$

$+ d^2u/(r^2\,\text{sine}^2\theta\, d\varphi^2) - 2dw/(r\,d\varphi)),$

$Y - dp/(\rho r\,d\theta) = r\,\partial v/\partial t + 2uv - r\,\text{sine}\,\theta\cos\theta\, w^2 - \nu(r\,d^2v/dr^2 +$

$4\,dv/dr + 2v/r + d^2v/(r\,d\theta^2) + 2\,du/(r^2 d\theta) + (dv/(r\,d\theta))\cot\theta -$

$v/(r\,\text{sine}^2\theta) + d^2v/(r\,\text{sine}^2\theta\, d\varphi^2) - 2\cos\theta\, dw/(r\,\text{sine}\,\theta\, d\varphi)),$

$Z - dp/(\rho\,\text{sine}\,\theta\, d\varphi) = r\,\text{sine}\,\theta\,\partial w/\partial t + 2\,\text{sine}\,\theta\, uw + 2r\cos\theta\, v\, w$

$- \nu(r\,\text{sine}\,\theta\, d^2w/dr^2 + 4\,\text{sine}\,\theta\, dw/dr + 2\,\text{sine}\,\theta\, w/r + \text{sine}\,\theta\, d^2w/(r\,d\theta^2)$

$+ 3\cos\theta\, dw/(r\,d\theta) - 2w\,\text{sine}\,\theta/r + d^2w/(r\,\text{sine}\,\theta\, d\varphi^2) + 2du/(r^2\,\text{sine}\,\theta\, d\varphi)$

$+ 2\cot\theta\, dv/(r\,\text{sine}\,\theta\, d\varphi))$

Colorado Springs, May 26, 1896--commenced Jan., 1895.

NOTE A.

For convenient reference the various equations are collected from the different parts of the foregoing paper; and a tabulated analysis of the terms appearing in the equations is added. Special attention is given to the terms in the equations referred to cylindrical and spherical coordinates.

EQUATIONS FOR RECTANGULAR COORDINATES

For a perfect liquid;

$$X - dp/(\rho dx) = \partial u/\partial t,$$

$$Y - dp/(\rho dy) = \partial v/\partial t,$$

$$Z - dp/(\rho dz) = \partial w/\partial t.$$

For a viscous liquid;

$$X - dp/(\rho dx) = \partial u/\partial t - (d^2u/dx^2 + d^2u/dy^2 + d^2u/dz^2),$$

$$Y - dp/(\rho dy) = \partial v/\partial t - (d^2v/dx^2 + d^2v/dy^2 + d^2v/dz^2),$$

$$Z - dp/(\rho dz) = \partial w/\partial t - (d^2w/dx^2 + d^2w/dy^2 + d^2w/dz^2).$$

Equation of continuity;

$$du/dx + dv/dy + dw/dz = 0.$$

EQUATIONS FOR CYLINDRICAL COORDINATES

For a perfect liquid;

$$X - \partial p/(\rho \partial r) = \partial u/\partial t - v^2 r,$$

$$Y - \partial p/(\rho r\, \partial\theta) = r\, \partial v/\partial t + 2uv,$$

$$Z - \partial p/(\rho \partial z) = \partial w/\partial t.$$

For a viscous liquid;

$$X - \partial p/(\rho \partial r) = \partial u/\partial t - rv^2 - \frac{\mu}{\rho}(\partial^2 u/\partial r^2 + \partial u/(r\, \partial r) - u/r^2 + \partial^2 u/(r^2\, \partial\theta^2)$$

$$- 2\, \partial v/(r\, \partial\theta) + \partial^2 u/\partial z^2),$$

$$Y - \partial p/(\rho r\, \partial\theta) = r\, \partial v/\partial t + 2uv - \frac{\mu}{\rho}(r\partial^2 v/\partial r^2 + 3\partial v/\partial r + \partial^2 v/(r\, \partial\theta^2)$$

$$+ 2\partial u/(r^2\, \partial\theta) + r\, \partial^2 v/\partial z^2),$$

$$Z - \partial p/(\rho \partial z) = \partial w/\partial t - \frac{\mu}{\rho}(\partial^2 w/\partial r^2 + \partial w/(r\, \partial r) + \partial^2 w/(r^2 \partial\theta^2) + \partial^2 w/\partial z^2).$$

Equation of continuity;

$$\partial(ur)/\partial r + r\, \partial v/\partial\theta + r\, \partial w/\partial z = C/$$

EQUATIONS FOR SPHERICAL COORDINATES

For a perfect liquid;

$$X - \partial p/(\rho \partial r) = \partial u/\partial t - rv^2 - r\, \sin^2\theta\, w^2,$$

$$Y - \partial p/(\rho r\, \partial\theta) = r\, \partial v/\partial t + 2uv - r\, \sin\theta\, \cos\theta\, w^2,$$

$Z - dp/(\rho r \sin\theta\, d\varphi) = r \sin\theta\, \partial w/\partial t + 2 \sin\theta\, u\, w + 2r \cos\theta\, v\, w.$

For a viscous liquid;

$X - dp/(\rho dr) = \partial u/\partial t - v^2 r - r \sin^2\theta\, w^2 - \mu/\rho(d^2u/dr^2 + 2du/(r\, dr) -$

$2u/r^2 + d^2u/(r^2 d\theta^2) + \cot\theta\, du/(r^2 d\theta) - 2dv/(r\, d\theta) - \cot\theta\, 2v/r +$

$d^2u/(r^2 \sin^2\theta\, d\varphi^2) - 2\, dw/(r\, d\varphi)),$

$Y - dp/(\rho r\, d\theta) = r\,\partial v/\partial t + 2uv - r \sin\theta \cos\theta\, w^2 - \mu/\rho(r\, d^2v/dr^2 +$

$4\, dv/dr + 2v/r + d^2v/(r\, d\theta^2) + 2du/(r^2 d\theta) + \cot\theta\, dv/(r\, d\theta) -$

$v/(r \sin^2\theta) + d^2v/(r \sin^2\theta\, d\varphi^2) - 2 \cos\theta\, dw/(r \sin\theta\, d\varphi)),$

$Z - dp/(\rho r \sin\theta\, d\varphi) = r \sin\theta\,\partial w/\partial t + 2\sin\theta\, uw + 2r \cos\theta\, vw -$

$\mu/\rho(r \sin\theta\, d^2w/dr^2 + 4 \sin\theta\, dw/dr + 2 \sin\theta\, w/r + \sin\theta\, d^2w/(r\, d\theta^2) +$

$3\cos\theta\, dw/(r\, d\theta) - 2w \sin\theta/r + d^2w/(r \sin\theta\, d\varphi^2) + 2du/(r^2 \sin\theta\, d\varphi)$

$+ 2 \cot\theta\, dv/(r \sin\theta\, d\varphi)).$

Equation of continuity;

$d(ur^2)/(r^2 dr) \sin\theta + d(v \sin\theta)/d\theta + \sin\theta\, dw/d\varphi = 0.$

AN ANALYSIS OF THE EQUATIONS

Rectangular coordinates:

The terms found in the equations when referred to rec-

tangular coordinates need but little explanation. The terms

$$d^2u/dx^2, d^2u/dy^2, \text{and } d^2u/dz^2$$

are due to the variation in the velocity u in the directions x,y/and z respectively. And the terms found in the other equations are due to a similar variation in v and w.

Cylindrical coordinates:

In equations for a viscous liquid

in the direction of x-axis

$-r\,v^2$ results from centrifugal force due to v,

d^2u/dr^2 results from du/dr,

$du/(r\,dr) - u/r^2$ results from displacement u/r,

(see page ~~53~~ 58)

$d^2u/(r^2 d\theta^2)$ results from $du/d\theta$,

$-2\,dv/(r\,d\theta)$ results from centrifugal force $-rv^2$;

d^2u/dz^2 results from du/dz;

in the direction of y-axis

$2uv$ results from combined action of v and u (see p 58

$r\, d^2v/dr^2 + dv/dr$ results from dv/dr,

$2\, dv/dr$ .... $2\, u\, v$,

$d^2v/(r\, d\theta^2)$ .... $dv/d\theta$,

$2\, du/(r^2\, d\theta)$...................... $2\, u\, v$,

$r\, d^2v/dz^2$........................ dv/dz;

in the direction of the z-axis.

The equation for the forces acting in this direction is exactly the same as that for rectangular coordinates.

Spherical coordinates:

In the direction of the x-axis

$-v^2 r$ results from centrifugal force due to v.

$-r \sin^2\theta\, w^2$.................................. w,

d^2u/dr^2 results from du/dr,

$2\, du/(r\, dr) - 2u/r^2$............ $2u/r$ (see page 53),

$d^2u/(r^2\, d\theta^2) + \cot\theta\, du/(r^2 d\theta)$.... $du/d\theta$,

$-2dv/(r\, d\theta) - \cot\theta\, 2v/r$....... centrifugal force $-rv^2$,

$d^2u/(r^2 \sin^2\theta\, d\varphi^2)$................ $du/d\varphi$,

$-2dw/(r\ d\varphi)$....................centrifugal force $-r\ w^2\sin^2\theta$ ~~$-rw\ \theta$~~

in the direction of the y-axis

$2\ u\ v$ ....combined action of u&v,

$-r\ \sin\theta\cos\theta\ w^2$................centrifugal force due to w,

$r\ d^2v/dr^2 + 2dv/dr$ ....dv/dr,

$2dv/dr + 2v/r$ ....$2\ u\ v$,

$d^2v/(r\ d\theta^2)$ ....$dv/d\theta$,

$2\ du/(r^2\ d\theta)$...................$2\ u\ v$,

$\cot\theta\ dv/(r\ d\theta) - v/(r\ \sin^2\theta)$...$v\ \cot\theta$ (see page 53),

$d^2v/(r\ \sin^2\theta\ d\varphi^2)$ ....$dv/d\varphi$,

$-2\ \cos\theta\ dw/(r\ \sin\theta\ d\varphi)$.......centrifugal force $-r\ \sin\theta\ \cos\theta w^2$;

in the direction of the z-axis

$2\sin\theta\ w\ u$ due to combined action of u&w(see p.37),

$2r\ \cos\theta\ v\ w$ ....v&w(see p.39),

$r\ \sin\theta\ d^2w/dr^2 + 2\sin\theta\ dw/dr$....$dw/dr$,

$2\sin\theta\ dw/dr + 2\sin\theta\ w/r$......$2\sin\theta\ w\ u$,

$\sin\theta\ d^2w/(r\ d\theta^2) + \cos\theta\ dw/(r\ d\theta)$....$dw/d\theta$,

$2\cos\theta\ dw/(r\ d\theta) - \sin\theta\ 2w/r$........$2r\ \cos\theta\ v\ w$,

$d^2w/(r \sin\theta\, d\varphi^2)$.................$dw/d\varphi$,

$2\ du/(r^2 \sin\theta\ d\varphi)$...............$2\ \sin\theta\ w\ u$,

$2\ \cot\theta\ dv/(r\ \sin\theta\ d\varphi)$.........$2\ r\ \cos\theta\ v\ w$.

NOTE B

Advantages in the preceeding method of derivation.

The method of deriving the equations for a viscous liquid which has been received most favorably is that of Stokes. Both Lamb and Basset have used his method, the first in part and the second entirely, in their works on hydrodynamics.

But Stokes method is based upon two hypotheses.(see Introduction). The second hypothesis has not been received very favorably because it involves the assumption that the motion is slow.

The method which I have used is based upon the definition of the coefficient of viscosity;and involves no hypothesis save that of continuity of motion .

The derivation of the equations is much simpler by my method than by former methods unless an exception is made of the method of O. E. Meyer. The equations of Meyer are based upon an assumption which does not satisfactorily explain part

of the terms(see Introduction).

The development of the equations of motion by analysis for cylindrical and spherical coordinates enables us to give a definite meaning to each term in the equations and may aid materially in their solution.

The method definitely determines when the equations hold and when they do not as is shown in NOTE C.

The principal feature of this paper is the relation established between the motion of a perfect and a viscous liquid(see pages 37 to 47).

NOTE C

The first appreoximation which was made in this paper is found on pages 44 and 46 where all terms except the first are neglected in the expression

$$(1/dt)(du/u - du^2/u^2 + du^3/u^3 - du^4/u^4 \text{ etc.})$$

This approximation is admissible unless the acceleration in the velocity is very great which is not usually the case.

In the second place we suppose the rate of inflow over the face in the direction perpendicular to the velocity considered to be uniform along the normal to the face. It is easy to conceive that the inflow might be accelerated. But if it were the acceleration neglected would be of the second order and might reasonably be neglected.

If our conclusions are true the equations hold not only when the motion is slow but in all cases except where the motion is rapidly accelerated in which case we could not neglect ~~the~~ so many terms in the equation above.

NOTE D

On the motion of a liquid in which a circular cylinder of infinite length is slowly rotating.

Whatever motion takes place in the liquid is due to internal friction and is two dimentional, i.e. in the plane perpendicular to the axis of the cylinder.

Let the very natural supposition be made that the layer of liquid contiguous to the surface of the cylinder moves with the surface . .Then the concentric layers will have a motion due to internal friction. There will be no motion parallel to the axis of the cylinder, consequently $w = C$: the only force acting normal to the cylinder is that due to the centrifugal force but since the velocity of rotation is small the products and squares of the velocity may be neglected without sensible error consequently $u = C$ also. Therefore to completely determine the motion of the liquid we need consider only the second of the equations for cylindrical coordi-

nates.

Since the initial condition was one of rest the conditions of equilibrium demand that

$$\partial v/\partial t = \frac{\mu}{\rho}(r\, d^2v/dr^2 + 3\, dv/dr +$$

$2du/(r^2\, d\theta)$ disappearing since $u = 0$,

$r\, d^2v/dz^2$ disappearing since the motion is two dimentional,

$d^2v/(r\, d\theta^2)$ disappearing since the motion is symetrical with respect to the axis of the cylinder,

Since the angular velocity of rotation is small

$\partial v/\partial t$ becomes dv/dt;

and as soon as the motion becomes steady

$$dv/dt = 0.$$

So that when the angular velocity is constant the motion of the liquid is determined from

$$r\, d^2v/dr^2 + 3\, dv/dr = 0.$$

Solving

$$r\,v = K\,r + C/r.$$

At an infinite distance $r\,v$ is finite consequently

$$K = 0,$$

$$r\,v = C/r.$$

Let ω be the angular velocity of the cylinder, then

$$C = a^2\omega,$$

$$r\,v = a^2\omega/r,$$

where a is the radius of the cylinder.

But if we consider the cylinder as filled with the liquid and consider the motion of the inclosed liquid

$$C = 0,$$

$$a\omega = K\,a,$$

$$K = \omega,$$

$$r\,v = r\,\omega.$$

Hence the inclosed liquid moves as a solid body.

(See Basset Vol.2 ,page 282)

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Approved

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